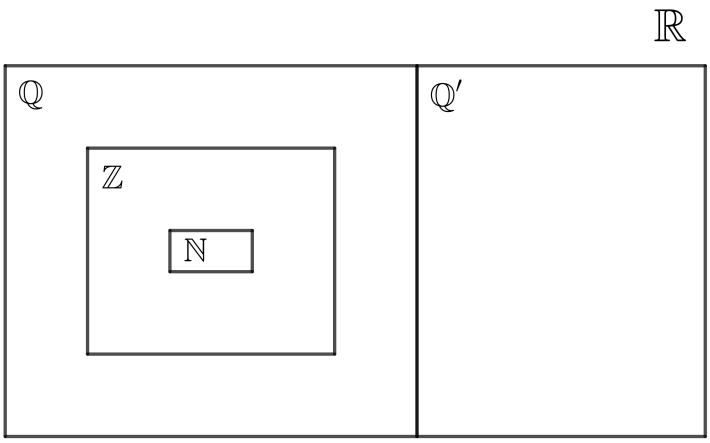
1.5. Irrational numbers

1. `Definition`
   Irrational numbers these are numbers that **cannot be expressed** as a fraction, for example $\sqrt{2}$. π, e, ϕ $\sqrt[3]{3}, \ldots$

   The set of rational numbers - $\mathbb{Q}$ and the set of irrational numbers - $\mathbb{Q}'$ are **mutually exlusive**. It means that they have no common elements.

2. `The structure of real numbers`



3. `How to prove that a number` $\sqrt{p}$ `is irrational, when` p `is a prime number?`
   As an example lets consider the following problem: *Prove, that* $\sqrt{2}$ *is irrational number.*

   Proof (by contradiction): Lets assume that $\sqrt{2}$ is a rational number. Then

$$\sqrt{2} = \frac{a}{b}, \tag{1}$$

   where $a, b \in \mathbb{Z}$, $b \neq 0$. Assume also that a and b are relatively prime. If we square both sides of the equation (1), we have

$$2 = \frac{a^2}{b^2}. \tag{2}$$

   It means that $2a^2 = b^2$. Notice, that b^2 is an even number, therefore b is an even number. Meaning, that on the right-hand side of the equation (2) there must be an even number of factor 2. But on the left-hand side there is only one factor of 2, because $\gcd(a, b) = 1$. Hence our initial assuption was false, which means that $\sqrt{2}$ is irrational number. □

EXERCISES

1. Write down all irrational numbers from the set $\left\{4, -\frac{1}{3}, \frac{\sqrt{3}}{2}, 0, 0.75, 0.(31), 3\pi, -2\sqrt{7}\right\}$

2. Prove that:

   (a) $\sqrt{3}$ (b) $\sqrt{5}+2$ (c) $\sqrt{2}-4$ (d) $\frac{\sqrt{2}+1}{3}$

   is irrational number.

3. Find two irrational numbers that are between

   (a) $\sqrt{5}$ and $\sqrt{7}$ (b) $-\sqrt{3}$ and $-\sqrt{2}$ (c) $\frac{\sqrt{6}}{2}$ and $\frac{\sqrt{6}}{4}$

4. Decide if the following statements are true or false. Select 'T' if the statement is true, or 'F' if it is false.

   (a) Every rational number is also an integer number. T F

   (b) Every natural number is also a rational number. T F

   (c) Every natural number is also a whole number. T F

   (d) Every whole number is also a natural number. T F

   (e) Every real number is an irrational number. T F